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CONDITIONS ALONG THE BOUNDARIES OF BLADED ZONES WITHIN THE FLOW TRACTS OF TURBINES

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SUMMARY

Hamilton's variational principle is applied to derive a system of conditions which expresses the balance of momentum and energy of an ideal gas across the selvadges* of bladed zones within the flow tracts of turbines. This system provides the background for a correct formulation of optimal design problems for turbines and compressors. The exposition follows the model of a large number of blades when the basic equations can be averaged over the azimuthal co-ordinate.

An analysis is given of the obtained conditions and a computational algorithm described.

KEY WORDS Turbines Compressors Optimal Design Blade Design

INTRODUCTION

In the design of turbomachines, it has become traditional to consider two main approaches. The first is connected with calculation of flow under the assumption that the shape of the blades is prescribed (direct problem). The second requires determination of this shape for some given characteristics of a flow (inverse problem). Both approaches contain a factor which is usually thought of as prescribed by the designer according to experimental tests, analysis of prototypes, intuitive considerations etc. The designer commonly endeavours to suggest a turbine possessing an extremal value of some integral characteristic, e.g. its efficiency.

Turbines now in action are already very effective, but further increases of efficiency are still very important because of their high power levels.

Engineering recommendations obtained to date do not guarantee that more efficient designs do not exist. The complete solution to the problem may be obtained by the technique of optimal control theory. The controlling factor which was earlier being prescribed by a designer according to the considerations mentioned above would then be provided by only the requirement of maximality of the efficiency.

The realization of this technique for an axisymmetric flow model within a turbine encounters the difficulty connected with the absence of continuity conditions across the boundaries dividing the zones of blades from those of a free stream within the flow tract. These conditions represent a necessary part of the analysis of the flow itself; in their absence, it is impossible to write down the corresponding conditions for the adjoint variables, that is to formulate the problem of optimal design of a flow tract itself.

* The reviewer of this paper confesses to be unfamiliar with the term 'selvadge(s)' which appears frequently herein. Clearly, however, it (variously) means 'boundary', 'edge', 'projection of edge line' (meridional or axial), 'locus of leading edges', etc.

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In what follows, we give the derivation of the mentioned system of conditions for a flow of an ideal gas through a turbine posessing an infinite number of blades.

Traditional conditions across the boundaries dividing different regions of a flow tract[1–3] usually suggest continuity of a meridional component of velocity. This is physically quite natural so long as we consider flows around some finite number of blades. However, in close vicinity of the boundaries of zones occupied by blades high velocity gradients arise because the flow has to change direction rapidly. Within the framework of the Lorentz scheme which is known to be connected with the infinitely dense system of blades, the regions of high gradients are imitated by the surfaces across which velocity suffers jumps.† For this scheme, the conservation laws do not necessitate continuity of the velocity vector. We give in what follows the derivation of the conditions which express the balance of momentum and energy across the boundaries. The procedure is based upon Hamilton's variational principle for the case of an ideal gas. The obtained conditions are discussed together with a corresponding computational algorithm. Alternative derivation of the kinematical condition was given in Reference 4 for the case of incompressible fluid; the question was also considered in Reference 5.

THE VARIATIONAL PRINCIPLE

The variational description of a flow of an ideal gas has been given in much detail by Herival[6] (see also Reference 7). If the fluid particles are compelled to move along some prescribed family of surfaces, the variational principle is modified according to Hellinger.[8] In References 7 and 8, however, no conditions were obtained along the surfaces dividing the regions where the motion of a fluid is described by different sets of equations.

According to the mentioned principle, the functional

$$I=\int_0^{t_k}\mathrm{d}t\int_A\left[\tfrac{1}{2}\rho\mathbf{c}^2-\rho(U+V)\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{1}$$

possesses a stationary value under the additional constraints

$$\left.\begin{aligned}&\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\mathbf{c}=0\\&\frac{\partial S}{\partial t}+\mathbf{c}\cdot\operatorname{grad}S=0\\&\frac{\partial\chi}{\partial t}+\mathbf{c}\cdot\operatorname{grad}\chi=0\end{aligned}\right\} \tag{2}$$

Here A denotes the moving fluid volume, $\mathbf{c}$ the absolute velocity vector, ρ the density, S the entropy, $U=U(\rho, S)$ the internal energy and $V=V(x, y, z, t)$ the potential of the external forces. Within the Lorentz scheme, which will be followed throughout the paper, the flow surfaces are considered as coincident with those of the blades, both being represented by the equation $\chi(x, y, z, t)=0$ in some fixed laboratory co-ordinate system. In cylindrical

† A trivial example: the rotational flow (momentum of azimuthal velocity $Rc_u\neq 0$) entering a system of fixed rectilinear blades where $Rc_u=0$ according to the boundary conditions. The jump of Rc_u across the front boundary is accompanied by an impulsive reaction of the blades upon the flow. Here R is the radius and c_u is the azimuthal component of velocity.

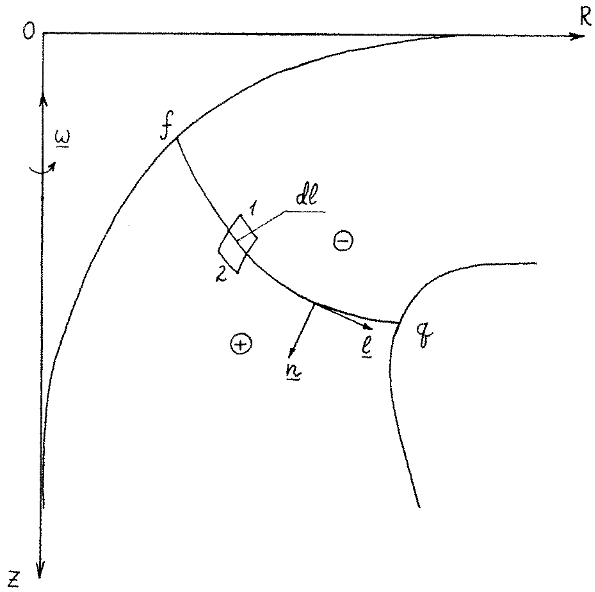

Figure 1. Meridional projection of the surface dividing the zone of blades from that of a free stream

co-ordinates R, φ, z, this last equation may be represented as ($\mathbf{r} \equiv \mathbf{r}(x, y, z)$)

$$\chi(r, t) = \varphi - \Phi(R, z) - \omega t = 0 \tag{3}$$

where ω designates the angular velocity of the shaft.

Note that the last of equations (2) is prescribed only within the zone of blades ((+)-zone, Figure 1). This zone is assumed to be divided from the free stream region ((−)-zone, Figure 1) by the surface of rotation

$$F(\mathbf{r}) = 0 \tag{4}$$

The intersection of this surface with a meridional plane is represented by the curve fq (Figure 1).

Let $\mathbf{a}(a, b, c)$ be the Lagrange co-ordinates of a fluid particle defined as the initial values of its rectangular co-ordinates x, y, z. Let further $t^* = t^*(\mathbf{a})$ be the time at which the particle $\mathbf{a}(a, b, c)$ intersects the boundary between the (+) and (−) regions. Equations (1) and (2) can then be rewritten in the form (where the dot denotes differentiation with respect to t for fixed values of $\mathbf{a}$)

$$I = \int_0^{t_k} \mathrm{d}t \int_A [\tfrac{1}{2}\rho(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \rho(U + V)]T \,\mathrm{d}\mathbf{a}; \qquad T = \frac{\mathscr{D}(x, y, z)}{\mathscr{D}(a, b, c)};$$

$$T\rho = T\rho(\mathbf{a}, t) = T\rho(\mathbf{a}, 0) \equiv \rho_0, \qquad 0 \leqslant t \leqslant t_k;$$

$$S = S(\mathbf{a}, t) = S(\mathbf{a}, 0) \equiv S^-, \qquad 0 \leqslant t < t^*;$$

$$S = S(\mathbf{a}, t) = S(\mathbf{a}, t^*) \equiv S^+, \qquad t^* \leqslant t \leqslant t_k;$$

$$\chi(r(\mathbf{a}, t), t) = \chi(r(\mathbf{a}, t^*), t^*) \equiv \chi^+, \qquad t^* \leqslant t \leqslant t_k.$$

NECESSARY CONDITIONS OF STATIONARITY

With the aid of the Lagrange multipliers α, β, γ we construct an augmented functional

$$\int_0^{t_k} \mathrm{d}t \int_A \{[\tfrac{1}{2}\rho(\dot{x}^2+\dot{y}^2+\dot{z}^2)-\rho(U+V)]T-\alpha[T\rho(\mathbf{a},t)-\rho_0]\}\,\mathrm{d}\mathbf{a}$$
$$-\int_0^{t^*}\mathrm{d}t\int_A [S(\mathbf{a},t)-S^-]\rho\beta T\,\mathrm{d}\mathbf{a}-\int_{t^*}^{t_k}\mathrm{d}t\int_A \{[S(\mathbf{a},t)-S^+]\rho\beta+\rho\gamma[\chi(r(\mathbf{a},t),t)-\chi^+]\}T\,\mathrm{d}\mathbf{a}$$

and require its first variation to vanish, considering $x(\mathbf{a},t)$, $y(\mathbf{a},t)$, $z(\mathbf{a},t)$, $\rho(\mathbf{a},t)$, $S(\mathbf{a},t)$ as independent functional arguments.

The Euler equations now take the form

$$\delta\rho:\ T\left[\tfrac{1}{2}\mathbf{c}^2-(U+V+\alpha)-\rho\frac{\partial U}{\partial\rho}\bigg|_S\right]=0$$

$$\delta S:\ \rho T\left[\frac{\partial U}{\partial S}\bigg|_\rho+\beta\right]=0$$

$$\delta x:\ \frac{\partial}{\partial t}(\rho T\dot{x})+\sum_{a,b,c}\frac{\partial}{\partial a}\left[\frac{\partial T}{\partial x_a}\rho(\tfrac{1}{2}\mathbf{c}^2-U-V-\alpha)\right]$$
$$+\rho T\frac{\partial V}{\partial x}+\rho T\gamma\frac{\partial\chi}{\partial x}=0$$

and analogous equations corresponding to δy, δz. In view of the thermodynamical relations (p—pressure, Θ—temperature

$$\frac{\partial U}{\partial\rho}\bigg|_S=\frac{p}{\rho^2},\qquad \frac{\partial U}{\partial S}\bigg|_\rho=\Theta$$

the first two equations become

$$\tfrac{1}{2}\mathbf{c}^2-(U+V+\alpha)=\frac{p}{\rho},\qquad \beta+\Theta=0$$

The equations corresponding to δx, δy, δz, can simply be transformed to

$$\frac{\mathbf{dc}}{dt}=-\frac{1}{\rho}\,\mathrm{grad}\,p-\mathrm{grad}\,V-\gamma\,\mathrm{grad}\,\chi \tag{5}$$

The last term on the right-hand side expresses the reaction upon the moving gas from the blades which perform the prescribed motion.

If external forces are absent and if the motion is stationary and axisymmetric, then the projection of (5) along the azimuthal direction provides the relation $\gamma=-\mathrm{d}Rc_u/\mathrm{d}t$, where c_u is the φ-projection of the velocity $\mathbf{c}$.

The conditions along the boundary surfaces arise from surface integrals in the t, a, b, c-space in the expression for the first variation of the augmented functional. The corresponding terms are

$$\int_\Sigma \{p[(\mathcal{D}_{x_a}\delta x+\mathcal{D}_{y_a}\delta y+\mathcal{D}_{z_a}\delta z)\cos Na+(\mathcal{D}_{x_b}\delta x$$
$$+\mathcal{D}_{y_b}\delta y+\mathcal{D}_{z_b}\delta z)\cos Nb+(\mathcal{D}_{x_c}\delta x+\mathcal{D}_{y_c}\delta y+\mathcal{D}_{z_c}\delta z)\cos Nc]$$
$$+\rho_0(\dot{x}\,\delta x+\dot{y}\,\delta y+\dot{z}\,\delta z)\cos Nt+\rho_0(\tfrac{1}{2}\mathbf{c}^2-U-V)\cos Nt\,\delta t^*\}_-^+\,\mathrm{d}\Sigma \tag{6}$$

Here Σ denotes the boundary surface in the four-dimensional space t, a, b, c given by $t = t^*(a, b, c)$, $\mathscr{D}_{x_a}, \ldots$ denote the algebraic adjuncts of the elements $x_a, \ldots$ in the Jacobian T; $\cos Na, \ldots$ are direction cosines of $\mathbf{N}$ relative to the axes $a, \ldots$ in the four-dimensional space. Observe that the dividing surface is fixed in the Euler space where it is represented by equation (4). Introducing the Lagrange co-ordinates and differentiating over t, we obtain

$$F_a\dot{a} + F_b\dot{b} + F_c\dot{c} + F_t|_{\mathbf{a}} = 0$$

where

$$F_a = F_x x_a + F_y y_a + F_z z_a$$

etc.

It would be convenient to introduce local co-ordinates a, b, c along the Σ-surface such that at time $t = t^*(\mathbf{a})$ the relationship $F_b = F_c = 0$ were valid; then it becomes evident that

$$\frac{T}{F_a} = \frac{\mathscr{D}_{x_a}}{F_x} = \frac{\mathscr{D}_{y_a}}{F_y} = \frac{\mathscr{D}_{z_a}}{F_z} = -\frac{T\dot{a}}{F_t} \tag{7}$$

and

$$\left.\begin{aligned} &\cos Na = F_a/\sqrt{(F_a^2 + F_t^2)}, \qquad \cos Nb = \cos Nc = 0,\\ &\cos Nt = F_t/\sqrt{(F_a^2 + F_t^2)} \end{aligned}\right\} \tag{8}$$

The coefficient of δx in (6) in view of (7) and (8) is

$$\left\{\frac{T}{F_a\sqrt{(1+\dot{a}^2)}}[pF_x + \rho\dot{x}F_t]\right\}_-^+ \tag{9}$$

The factor before the square brackets is continuous across Σ because the same is true (see (7)) for F_x and $\mathscr{D}_{x_a} = y_b z_c - z_b y_c$ (the last due to the choice of the co-ordinates a, b, c); $\dot{a} = -F_t/F_a$ represents the velocity of the three-dimensional surface—the projection of Σ onto the three-dimensional space a, b, c. The result is the same for the coefficients of δy and δz.

The complete variation Δx at the Σ-surface is connected with the variations $\delta x, \delta t^*$ by the relationship

$$\Delta x = \delta x + \dot{x}\,\delta t^* \tag{10}$$

and the same for $\Delta y, \Delta z$. The complete variations $\Delta x, \Delta y, \Delta z$ are continuous across Σ:

$$[\Delta x]_-^+ = [\Delta y]_-^+ = [\Delta z]_-^+ = 0 \tag{11}$$

The (+)-limiting values $\Delta x_+, \Delta y_+, \Delta z_+$ from the zone of blades (Figure 1) are subject to the constraints

$$\left.\begin{aligned} &\chi_x\,\Delta x + \chi_y\,\Delta y + \chi_z\,\Delta z + \chi_t|_{\mathbf{r}}\,\delta t^* = 0\\ &F_x\,\Delta x + F_y\,\Delta y + F_z\,\Delta z = 0, \end{aligned}\right\} \tag{12}$$

the first of which express the condition that fluid cannot pass through a blade and the second is the matching conditions across the locus of leading edges. To take the latter constraints into account we introduce the additional Lagrange multipliers $\tilde{\lambda}T/F_a\sqrt{(1+\dot{a}^2)}$, $\tilde{\mu}T/F_a\sqrt{(1+\dot{a}^2)}$ and set the coefficients of $\Delta x, \Delta y, \Delta z$ in (6) equal to zero, taking care of equations (9)–(11) ($\nabla F \equiv \operatorname{grad} F$)

$$[p\,\nabla F + \rho\mathbf{c}F_t]_-^+ + \tilde{\lambda}\,\nabla\chi + \tilde{\mu}\,\nabla F = 0 \tag{13}$$

With the aid of

$$F_t = \frac{\partial F}{\partial t}\bigg|_{\mathbf{a}} = \frac{\partial F}{\partial t}\bigg|_{\mathbf{r}} + \nabla F \cdot \mathbf{c} = \nabla F \cdot \mathbf{c}; \qquad \nabla F = \mathbf{n} \cdot |\nabla F| \tag{14}$$

(here $\mathbf{n}$ denotes a normal to the meridional projection of the Σ-surface;† see Figure 1), we transform equation (13) to the form

$$[p\mathbf{n} + \rho c_n \mathbf{c}]_-^+ + \lambda\, \nabla\chi + \mu\mathbf{n} = 0 \tag{15}$$

where

$$c_n = \mathbf{c} \cdot \mathbf{n}, \qquad \lambda = \tilde{\lambda}/|\nabla F|, \qquad \mu = \tilde{\mu}/|\nabla F| \tag{16}$$

Owing to equations (10) and (12) the coefficient of δt^* in equation (6) is

$$[-p(\mathscr{D}_{x_a}\dot{x} + \mathscr{D}_{y_a}\dot{y} + \mathscr{D}_{z_a}\dot{z}) \cos Na - \rho_0(\tfrac{1}{2}\mathbf{c}^2 + U + V) \cos Nt]_-^+ + \frac{T\tilde{\lambda}}{F_a\sqrt{(1+\dot{a}^2)}} \frac{\partial \chi}{\partial t}\bigg|_{\mathbf{r}}$$

Taking equations (7) and (8) into account and keeping in mind that $\partial\chi/\partial t|_{\mathbf{r}} = -\omega$ (see equation (3)), we transform this last expression to

$$\frac{\rho_0 \dot{a}}{\sqrt{(1+\dot{a}^2)}} \left[\tfrac{1}{2}\mathbf{c}^2 + U + V + \frac{p}{\rho}\right]_-^+ - \frac{T\omega\tilde{\lambda}}{F_a\sqrt{(1+\dot{a}^2)}} \tag{17}$$

Projection of equation (13) along the unit vector $\mathbf{i}_\varphi$ determines the multiplier $\tilde{\lambda}$:

$$\tilde{\lambda} = -[\rho F_t \cdot Rc_u]_-^+ \tag{18}$$

Substituting this expression into (17), we arrive at the condition

$$\left[\tfrac{1}{2}\mathbf{c}^2 + U + V + \frac{p}{\rho} - \omega Rc_u\right]_-^+ = 0 \tag{19}$$

which expresses continuity of the enthalpy of the relative motion across the boundary Σ between (+) and (−) regions of the flow (the potential V of external forces is assumed continuous across Σ). For the case of incompressible flow, equation (19) demonstrates continuity of energy in the relative motion.

Conservation of mass of a fluid particle across Σ means that‡

$$[\rho c_n]_-^+ = 0 \tag{20}$$

The (+)-limiting values of the hydrodynamic variables are subject to the condition (see (3)

† Here and below we shall designate by Σ the dividing surface (4) in Eulerian space.

‡ The variational principle (1)–(2) was based on the assumption that the volume element of a fluid particle equals $dx\,dy\,dz$; to take the volume of the blades into account, we suggest that the fluid particle occupies some smaller volume $\tau\,dx\,dy\,dz$, $\tau \leqslant 1$. The parameter τ will be treated as a known function of the co-ordinates. The variational principle will now be modified as follows: the first of equations (2) takes the form

$$\partial(\rho\tau)/\partial t + \operatorname{div} \rho\tau\mathbf{c} = 0 \tag{2'}$$

and the volume element $dx\,dy\,dz$ in equation (1) is substituted by $\tau\,dx\,dy\,dz$. The equations of motion and the conditions along the dividing surface are derived along the lines outlined above. Equations (5) and (19) still hold, equations (15), (20) are modified to

$$[p\tau n + \rho\tau c_n \mathbf{c}]_-^+ + \lambda\, \nabla\chi + \mu\mathbf{n} = 0 \tag{15'}$$

$$[\rho\tau c_n]_-^+ = 0 \tag{20'}$$

and the last of equations (2))

$$\frac{d\chi}{dt} = -\omega + \nabla\chi \cdot \mathbf{c}^+ = -\omega + \frac{1}{R} c_u^+ - \frac{\partial\Phi}{\partial n} c_n^+ - \frac{\partial\Phi}{\partial l} c_l^+ = 0 \tag{21}$$

which expresses the requirement that the fluid particles follow the surface of a blade (i.e. not penetrate it.) Here $c_l = \mathbf{c} \cdot \mathbf{l}$ denotes the component of the velocity vector along the unit vector tangent to Σ in the meridional plane (see Figure 1); the function $\varphi = \Phi(R, z)$ describes the surface of a blade in a co-ordinate system rotating with the angular velocity ω.

Equation (15), (19) and (20) represent the complete system of conditions valid along the surface dividing the zone of blades from that of a free stream.† For the case of an incompressible fluid, this system is complete; for a compressible fluid, it should be complemented by some additional prescriptions, concerning e.g. the forces acting upon a fluid particle on Σ.

DISCUSSION

Eliminating λ and μ from equation (15), we obtain

$$[\mathbf{c}]_-^+ \cdot (\mathbf{n} \times \nabla\chi) = 0$$

or transforming the vector product,

$$[c_l]^+ + \frac{\partial\Phi}{\partial l} [Rc_u]_-^+ = 0 \tag{22}$$

The jump $[c_l]_-^+$ characterizes the discontinuity of the meridional velocity at the points of Σ.‡ It is also seen from equation (22) that the discontinuity disappears if the selvadge of a blade is disposed within the meridional plane: $\partial\Phi/\partial l = 0$; the jump $[c_l]_-^+$ also vanishes when the azimuthal velocity c_u is continuous across $\Sigma : [c_u]_-^+ = 0$.§

To illustrate the physical meaning of parameters λ and μ, we consider an elementary fluid volume dO in the vicinity of the entrance selvadge, this volume being restricted by two close cross-sections 1 and 2 and two close streamlines separated by the distance dl. Applying conservation of momentum in the Euler form,[9] we find that the blades produce the reactive inertial force d$\mathbf{Q}$ upon the fluid volume dO, this force being equal to (assume grad $V \equiv 0$, $\tau = 1$)

$$d\mathbf{Q} = [p\mathbf{n} + \rho c_n \mathbf{c}]_-^+ \, d\Sigma \tag{23}$$

where $d\Sigma = 2\pi R \, dl$. Comparison with equation (15) shows that

$$d\mathbf{Q} = -(\lambda \nabla\chi + \mu\mathbf{n}) \, d\Sigma \tag{24}$$

and the reactive force applied to dO along the selvadge turns out to be the sum of two forces: the force $-\lambda \nabla\chi \, d\Sigma$ which is normal to the surface of a blade, and the force $-\mu\mathbf{n} \, d\Sigma$

† In the absence of blades ($\lambda = \mu = 0$), the obtained conditions express the traditional conservation laws across shock waves.

‡ If we introduce the parameter τ (see comment in an earlier footnote), then equation (22) still holds.

§ In Reference 4, equation (22) was obtained due to formal consideration of δ-singularities of derivatives of the velocity in the equations describing axisymmetric incompressible flow within the zone of blades.

disposed within the meridional plane. The term $-\lambda\,\nabla\chi\,\mathrm{d}\Sigma$ expresses the reaction from the prescribed surface of a blade upon the fluid particle crossing Σ. The value of λ in accordance with equations (14), (16) and (18) is given by

$$\lambda = -\rho c_n [Rc_u]_-^+ \tag{25}$$

The term $-\mu n\,\mathrm{d}\Sigma$ in equation (24) represents the reactive force produced by the selvadge itself, this force being applied to fix the position of the Σ-surface in space. To determine the value of μ, we project equation (15) along the n-direction; in view of equations (3) and (25) we obtain

$$\mu = \frac{\partial\Phi}{\partial n}\,\rho c_n [Rc_u]_+^- + [p + \rho c_n^2]_+^-$$

For the incompressible case $\rho = \text{const.}$, $U = \text{const.}$, $[c_n]_-^+ = 0$; this last equation can be modified to (see equations (19), (21) and (22))

$$\mu = -\tfrac{1}{2}\rho([\mathbf{c}]_-^+)^2 \tag{26}$$

THE COMPUTATIONAL ALGORITHM

As an example we shall consider the numerical algorithm of the solution of the inverse problem for the incompressible flow in a turbine. It will be assumed that within the bladed zones, the field of angular momentum, Rc_u, is prescribed. The problem will be formulated as follows: given the flow rate ψ^*, the angular velocity ω of a shaft, the tangents $\mathrm{tg}\,\gamma|_{A\mathscr{D}} = c_z/c_R$ at the entrance and $\mathrm{tg}\,\gamma|_{BC} = c_z/c_R$ at the exit section of a turbine, the entrance distributions $E(\psi)|_{A\mathscr{D}}$, $Rc_u(\psi)|_{A\mathscr{D}}$ and the fields Rc_u of moments within the zones $\mathscr{D}_2$, $\mathscr{D}_4$ occupied by blades (Figure 2), determine the parameters of the flow together with the geometric characteristics of the blades. The flow is assumed axisymmetric, and the exit selvadges of the blades disposed within the meridional plane; the angular momentum Rc_u is also assumed continuous across the exit selvadges.

To construct the algorithm with the aid of equations (19), (20) and (22), we use some of the results of Reference 2, in which the problem of blade design for a radial-axial turbine in a rotational flow was considered, the meridional velocity component being assumed continuous.

We take the polar cylindrical co-ordinates r, θ (Figure 2) and introduce the stream function

$$c_\theta = -\frac{1}{R\tau}\frac{\partial\psi}{\partial r}, \qquad c_r = \frac{1}{rR\tau}\frac{\partial\psi}{\partial\theta}$$

where the parameter τ is determined as described in a footnote earlier in the paper.

Equations (2) and (5) are now rewritten in the form

$$\operatorname{div}\frac{1}{R^2\tau}\nabla\psi = F_1 \tag{27}$$

$$T(\psi, \Phi) = \frac{\tau r}{R}(Rc_u - \omega R^2) \tag{28}$$

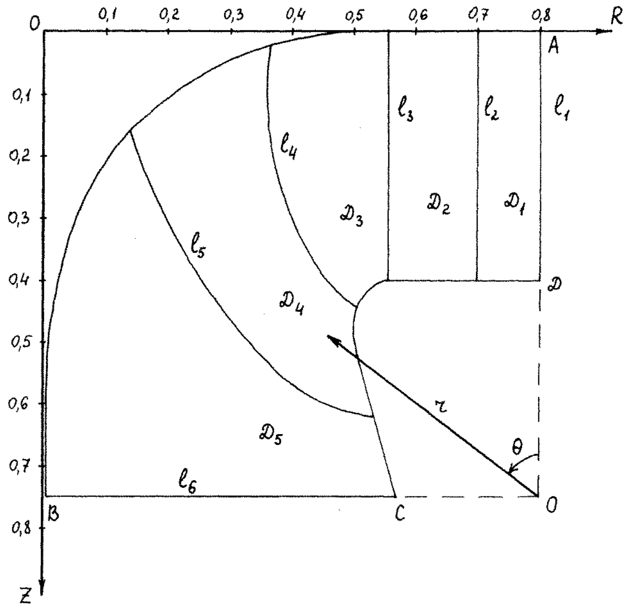

Figure 2. Meridional projection of a flow tract, $\mathscr{D}_1$ is a free zone (inlet), $\mathscr{D}_2$ is the rotor (or stator) zone, $\mathscr{D}_3$ is a free zone between the blades, $\mathscr{D}_4$ is the stator (or rotor) zone, and $\mathscr{D}_5$ is a free zone (wake)

where

$$T(f, g)=\frac{\mathscr{D}(f, g)}{\mathscr{D}(\theta, r)}$$

$$F_1=\begin{cases} F_1^-=\tau\left[\dfrac{dE}{d\psi}-\dfrac{1}{2R^2}\dfrac{d}{d\psi}(R^2c_u^2)\right], & \mathscr{D}_1, \mathscr{D}_3, \mathscr{D}_5 \\ F_1^+=\tau\dfrac{dI}{d\psi}+\dfrac{1}{rR}T(\Phi, Rc_u), & \mathscr{D}_2, \mathscr{D}_4 \end{cases} \tag{29}$$

Here $E(\psi)$ and $I(\psi)$ denote Bernoulli constants for absolute and relative motions respectively. The parameter τ will be assumed continuous and equal to unity at the selvadges of the blades.

We now introduce new independent variables ξ, η so as to transform the initial region into a unit square, the curvilinear selvadges going over to the sections of the straight lines $\xi=\text{const.}$ (Reference 2).

Nodes of the computational grid in the ξ, η-plane will be chosen as shown on Figure 3. In the close neighbourhood of the entrance selvadge ($\xi=\xi_l$) of the blades, consider two fictitious straight lines $\xi=\xi_{l+1}^-$ and $\xi=\xi_{l-1}^+$ which are formally coincident with $\xi=\xi_{l+1}$ and $\xi=\xi_{l-1}$. The line $\xi=\xi_{l+1}^-$ will be related to the free rotational flow zone, and the line $\xi=\xi_{l-1}^+$ to the zone of blades.

Equation (20) expresses continuity of the stream function across the entrance selvadges of the blades:

$$\psi_{l,j}^+=\psi_{l,j}^- \tag{30}$$

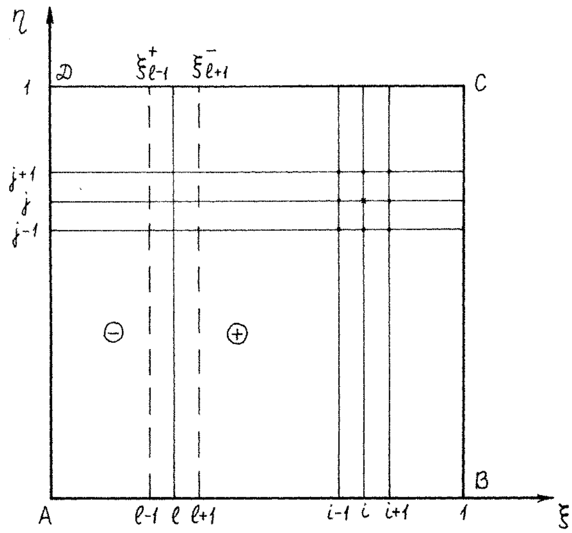

Figure 3. Construction of the computational scheme

At each inner node of the grid except the entrance selvadges, equation (27) is replaced by the difference equation in which derivatives are approximated by central difference quotients

$$\Delta_{ij} = \sum_{m=i-1}^{i+1} \sum_{n=j-1}^{j+1} Q_{m,n}\psi_{m,n} + (rRF_1)_{ij} = 0 \tag{31}$$

For the nodes disposed at the entrance selvadges, the corresponding difference equations will be

$$\begin{aligned} \Delta^-_{l,j} &= \sum_{m=l-1}^{l} \sum_{n=j-1}^{j+1} Q_{m,n}\psi_{m,n} + \sum_{n=j-1}^{j+1} Q_{l+1,n}\psi^-_{l+1,n} + (rRF^-_1)_{l,j} = 0 \\ \Delta^+_{l,j} &= \sum_{n=j-1}^{j+1} Q_{l-1,n}\psi^+_{l-1,n} + \sum_{m=l}^{l+1} \sum_{n=j-1}^{j+1} Q_{m,n}\psi_{m,n} + (rRF^+_1)_{l,j} = 0 \end{aligned} \tag{32}$$

Equation (22) is also approximated by a difference equation which is transformed to

$$\psi^-_{l+1,j} - \psi_{l-1,j} = \psi_{l+1,j} - \psi^+_{l-1,j} - G_{l,j} \tag{33}$$

where

$$G_{l,j} = 2h \left(\frac{\tau R r \dfrac{\partial \Phi}{\partial l} [Rc_u]^-_+}{\partial \xi / \partial \theta - \dfrac{\partial \theta_l}{\partial r} r^2 \dfrac{\partial \xi}{\partial r}} \right)_{l,j}$$

Here h denotes the difference step along the ξ-co-ordinate, and θ_l the polar angle θ at the entrance selvadge. Since the flow rate through the channel is prescribed, we have at the walls

$$\psi(\xi, 1) = \psi^*, \qquad \psi(\xi, 0) = 0 \tag{34}$$

Equations (31)–(33) together with the boundary conditions provide a complete system of linear algebraic equations. As in Reference 2, this system is solved by the iterative scheme

$$\psi_{i,j}^{p+1} = \psi_{i,j}^{p} - t\Delta_{ij}^{p},$$

where t is an iteration parameter. At the entrance selvadges, the procedure is modified as follows

$$\left.\begin{aligned}(\psi_{l,j}^{-})^{p+1} &= (\psi_{l,j}^{-})^{p} - t(\Delta_{l,j}^{-})^{p}\\ (\psi_{l,j}^{+})^{p+1} &= (\psi_{l,j}^{+})^{p} - t(\Delta_{l,j}^{+})^{p}\end{aligned}\right\} \tag{35}$$

Now, in view of equation (30), we obtain

$$\sum_{n=j-1}^{j+1} (Q_{l-1,n}\psi_{l-1,n} + Q_{l+1,n}\psi_{l+1,n}^{-}) + (rRF_{1}^{-})_{l,j} = \sum_{n=j-1}^{j+1} (Q_{l-1,n}\psi_{l-1,n}^{+} + Q_{l+1,n}\psi_{l+1,n}) + (rRF_{1}^{+})_{l,j} \tag{36}$$

Comparing equations (36) and (33) and taking equation (34) into account, we obtain

$$\left.\begin{aligned}&\sum_{n=j-1}^{j+1} (Q_{l+1,n} + Q_{l-1,n})\lambda_{l+1,n}^{-} = ([rRF_{1}]_{-}^{+})_{l,j} - \sum_{n=j-1}^{j+1} Q_{l-1,n}G_{l,n}\\ &\lambda^{-}(\xi_{l+1}^{-}, 1) = \lambda^{-}(\xi_{l+1}^{-}, 0) = 0\end{aligned}\right\} \tag{37}$$

where

$$\lambda_{l+1,j}^{-} = \psi_{l+1,j}^{-} - \psi_{l+1,j}$$

The algebraic system (37) is solved by a standard technique which provides the values of $\psi_{l+1,j}^{-}$, which are used in the iterative scheme (35).

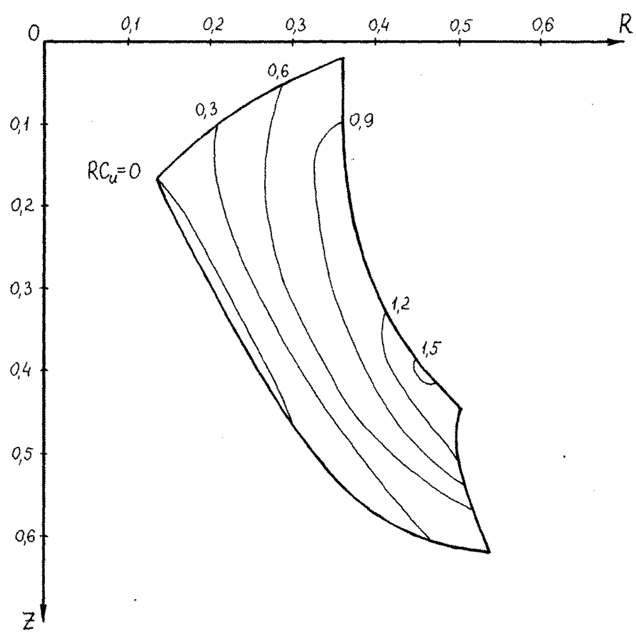

Figure 4. Distribution of the angular momentum Rc_u within the working shaft of a turbine

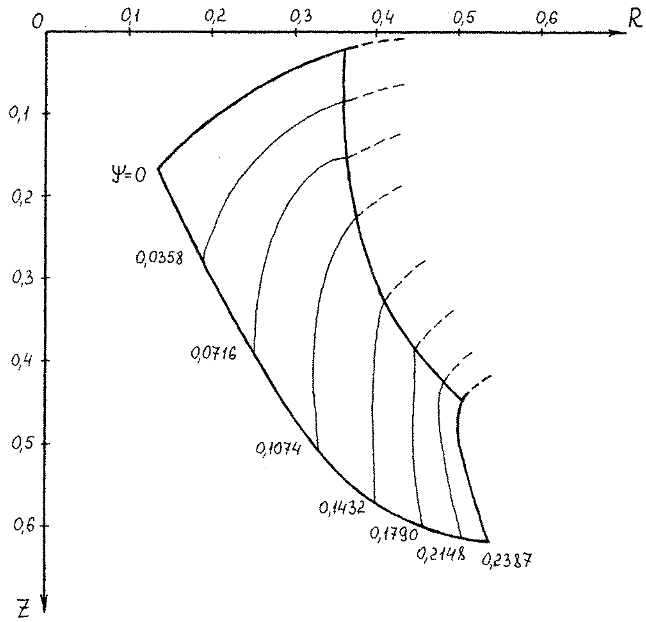

Figure 5. The lines $\psi = \text{const}$

The values of Φ are determined from equation (28) which is solved by the method of characteristics with the use of the boundary condition for Φ at the exit selvadges of the blades.[2]

The described algorithm was used to calculate the flow parameters within the flow tract of a turbine for the prescribed field of Rc_u shown in Figure 4. The inlet guide vanes were represented by prescribing the momentum Rc_u at the line $l_3: Rc_u(\psi)|_{l_3} = 0{\cdot}7451 + 2{\cdot}9954\psi$. The boundary conditions for the stream function at l_3 and l_6 were taken in the form $\text{tg}\,\gamma|_{l_3} = 0$, $\text{tg}\,\gamma|_{l_6} = 0{\cdot}5045\,(R - 0{\cdot}005)$; the parameter τ was set equal to unity. The results are shown in Figure 5 (the form of a flow tract shown in Figure 2), for $\psi^* = 0{\cdot}2387$, $\omega = 9{\cdot}95$, $E_0(\psi) = \text{Const}$.

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